The Fundamental Theorem of Algebra

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The equation that started it all: $x^2 + 1 = 0$

It once fell to my lot to teach Mathematics to first-year students at an Australian university. It is often said that you truly start to learn only when you start to teach. And so it was in my case.

In my blogs The Wonder That Is Pi, and The Two Most Important Numbers: Zero and One, I have introduced what I call the *number menagerie*. Different types of numbers were inducted into the art of reckoning or arithmetic, which later flowered into the science of quantity and symmetry called mathematics.[1]

The equation

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \sqrt{-1} \\ &= i. \end{aligned} \tag{1}$$

triggered the last great change, which led from the real numbers to the complex numbers.

In a manner of speaking, the one dimensional real numbers expanded into the two dimensional complex numbers. And why did it happen? Because of Equation (1). There is no real number whose square is negative. Hence, a new type of number having this property needed to be invented. It was granted an identifying tag i to show that it was *imaginary*.

I strongly recommend the unfamiliar reader to read my blog Expressions, Equations, and Formulae to get a firmer footing on what we discuss subsequently.

Vector Spaces and Fields

Most readers of this blog will be familiar with vectors as they arise in Physics. They are usually drawn on paper with two perpendicular coordinate axes. Vectors are shown as line segments with arrowheads: their lengths denote *magnitude*, and arrows *direction*. Vectors reside in vector spaces.

Fields are collections of numbers on which the four arithmetic operations apply. Division by zero, however, is excluded. The rational numbers, $\mathbb{Q}$, and real numbers $\mathbb{R}$ are fields. The natural numbers, $\mathbb{N}$, and integers, $\mathbb{Z}$ are not fields.

We are now at the threshold of the expansion of the real numbers, $\mathbb{R}$, into the complex numbers, $\mathbb{C}$. Both $\mathbb{R}$ and $\mathbb{C}$ are fields. The addition of the imaginary unit i might be seen as an ad hoc and

[1] Currently, mathematics is morphing into the science of patterns, and who knows what its dominant characteristic will become in the future?

opportunistic expansion of the real number field simply to accommodate solutions to $x^2 + 1 = 0$. But there is a deeper and wider logic to this expansion.

The addition of i has *extended* the solution space of the polynomial beyond $\mathbb{R}$. And this extension was necessitated by the quadratic equation Equation (1), which is a polynomial of *degree 2*. Adding i to $\mathbb{R}$ creates an *algebraic extension of degree 2*. This means the entire complex plane, $\mathbb{C}$, is simply an algebraic extension of degree 2, of $\mathbb{R}$. What does this mean?

A two-dimensional vector has components in two *orthgonal* directions, aligned with the x and y axes. Vectors are denoted as $a\mathbf{i} + b\mathbf{j}$ where a and b are real numbers and $\mathbf{i}$ and $\mathbf{j}$ the unit vectors in the x and y directions.

Likewise, a complex number may be written a $a(1) + b(i)$. Clearly, the real number 1 and the imaginary unit i function like the orthogonal unit vectors $\mathbf{i}$ and $\mathbf{j}$. So, we may identify $\mathbb{C}$ with a two-dimensional vector space. And pictorially, that is exactly what an Argand diagram is.

Can we multiply two complex numbers? Yes we can, as detailed in my blogs A Tetrad of Captivating Problems, and Demystifying Fractional Powers.

So, $\mathbb{C}$ is *both* a field and vector space.It is usual to associate a field with a vector space, as the scalar multipliers are supplied by the field and the vectors by the vector space. When we speak of $\mathbb{C}$ as the extension of $\mathbb{R}$ of degree 2 we are treating $\mathbb{R}$ as the field and $\mathbb{C}$ as the vector space.

Interestingly, this is denoted as $[\mathbb{C} : \mathbb{R}] = 2$, which means that any number in the new field may be built up as a linear combination of the two numbers, 1 and i, which is akin to building a new space with two basis vectors, 1 and i.

To summarize:

1. All number fields are also vector spaces.
2. When one number field, say X, is extended into a larger number field, K, the latter is also, by definition a field. Interestingly, K is also a vector space over the field X.
3. These field extensions are often restricted to allow the solution of specific polynomials.

Where will the number menagerie end?

The “holes” in the number line left by the integers were filled by the rational numbers. The remaining holes, like the number $\sqrt{2}$ were filled by the irrationals. When the real line was deemed complete, out popped Equation (1) to proclaim the inadequacy of the reals.

The thought that ran in my head was, “If the complex numbers well and truly account for the roots of polynomials like $x^2 + 1$, could there be some other type of polynomial that would require an even larger repertoire of numbers in order for its roots to be solved?

If white sheep are purely real numbers, and black sheep are complex numbers, with non-zero imaginary parts, could there be brown sheep, representing another type of number that might be needed to solve even more demanding polynomial equations? Would sheep of even more colours be needed to meet future challenges? This was the disquieting thought that ran in my mind as I prepared to teach mathematics to surveyors and engineers.

Dedekind's *schnitt* or cut

In the mid-1800s, there was a nagging question. How does one define an irrational number without recourse to geometry? If the whole numbers are all we confidently have—numbers we can touch and relate to—how can we define something like $\sqrt{2}$ without a geometric detour as the diagonal of a unit square?

I have given one solution to this conundrum in my blog How Are Numbers Built?. But the continued fraction expansions of numbers like $\sqrt{2}$ and e and π are not generalizable. And without a way for the general irrational number to be defined, how do we know *how many* irrationals there are, and what their *values* are?

Richard Dedekind pondered this question and came up with a solution that was partly geometric and partly set-theoretic.[2] The interested reader may consult Dedekind's original work in translation for the details [1].

In any case, once the real number line had been *completed* with a definition of the irrationals, the next examination of vacancies belonged in the realm of the complex numbers, where Gauss banished all thought of numbers "above" the complex with his Fundamental Theorem of Algebra.

Before we go into that, I want to make a small detour into something called the Intermediate Value Theorem.

The Intermediate Value Theorem

Gauss and the Fundamental Theorem of Algebra (FTA)

Galois' Bridge from Polynomials to Groups

DO WE SWAP THE LAST TWO SECTIONS?

Acknowledgements

Feedback

Please email me your comments and corrections.

A PDF version of this article is available for download here:

https://swanlotus.netlify.app/blogs/fundamental-theorem-of-algebra.pdf

References

[1] Dedekind, R. *Essays on the theory of numbers*. Dover Publications, 1963.

[2] I must confess to a certain unease with Dedekind's logic as, hidden somewhere inside it, is the assumption that the sets of rationals and irrationals are mutually exclusive sets, whose union is the reals. In other words, a dichotomy of the reals was assumed before the proof. Also, it is strange for a proof that claims to untether numbers from geometry to be based on a line of numbers which is "cut".

