Conceptual Abstract Algebra

R (Chandra) Chandrasekhar

2026-10-21 | 2026-10-29

Imagine...

Imagine you are in front of a house in a residential neighbourhood. The houses are predictable, the environment clean, the silence resounding, and the peace tangible. It is as if the suburb had morphed into a neighbourhood of heaven.

You ring the doorbell in front of a house. When the door is opened, you are aghast to find *inside the house* a tropical rain forest with its varied vegetation and wild animals. A cacophony of strange and scary sights and sounds.

Would you immediately run away, closing the door after you, or would you venture into the dark, dank depths of the uncharted forest inside?

Abstract Algebra, as a specialty of mathematics, has always seemed to me like that forbidding rain forest tucked away in a corner, behind the perfectly normal main doors of mathematics. One could live one's entire life without venturing past that threshold. And millions do.

This blog has been written to facilitate my entry into that rain forest after the initial trepidation has been overcome. And it is shared in the hope that it could help others make that momentous first step.

Music and Mathematics

Great music of any culture—Beethoven's *Ninth Symphony*, or Michio Miyagi's *Haru no Umi*, or Muthuswami Dikshitar's *Rangapura Vihara*—does not require expert musical knowledge to be appreciated. One only needs interest and a keen ear for melody and harmony, rhythm and beat, to enjoy these classics.

Likewise, to appreciate the beauty of mathematics one needs interest and an understanding of the logical principles and outcomes underlying mathematics. It should require no specialist preparation to be savoured and delighted in.

Alas that is not the case with contemporary mathematics. The maze of nomenclature, thickets of notation, and velvet curtains of symbols make natural appreciation all but impossible for the curious layperson. And I have always felt that the hoi polloi should not be fenced out by the literati.

The exquisite structures adorning the depths of mathematics are actually intricate masterpieces of beauty, no less than the greatest masterworks of art and music. Listening to great music should not require you to read musical notation. You only need interest and engagement. So too for mathematics.

It may be claimed that the innocent beauty of Creation itself is embedded in the warp and weft of mathematics. I have written this blog to de-mystify aspects of abstract algebra for myself. If, when they are shared they can do something similar form my readers, I would be deeply fulfilled.

Mathematics is self-regulating

Historically, mathematics has been marvellously self-regulated by her practitioners. Witness the revolution in arithmetic occasioned by the use of Hindu-Arabic numerals to replace Roman numbers. But sometimes, the practitioners race so far ahead that the general public gives up, as futile, the pursuit to keep up with changes.

Fun YT Video to watch first

If you can spare twenty minutes to watch a YouTube video, I will recommend this whimsical romp through the realm of mathematics created by Martin Kuppe, aka Zogg from Betelgeuse. In it, at around 11:30 there is an attractive map of a place playfully called "Mathematistan" [1]. Category gets a brief mention at around 14:45. By watching this video, you might just be able to retrieve the long lost feeling that mathematics once was fun.

XXXXXX Contact Kuppe for permission to use his map in this blog XXXXXX: email sent

In everyday life, we have nouns and verbs, names and actions, words and pictures. We will strive to look at these twin aspects as we survey the vast domain of mathematics.

Category Theory: Nouns and Verbs

Category Theory [2–7] (CT) is a relatively new branch of mathematics which aims to classify and connect the various branches of mathematics from within mathematics itself. As in everyday life, we have nouns and verbs.

> Objects (nouns) and morphisms (verbs) are the raw ingredients of Category Theory.

What the objects and morphisms are is not defined. This gives CT its "umbrella theory" mantle. It is a mathematics of mathematics or a theory of theories.

What qualifies as a Category?

For a collection of objects and morphisms to qualify as a Category, they must strictly obey the following three rules:

1. Identity: For every single object, there must exist a specific morphism that maps it perfectly to itself. This is the mathematical equivalent of a "do nothing" or "stay at home" action.
2. Composition: If a morphism takes you from object A to object B, and another morphism takes you from object B to object C, there must exist a valid, combined morphism that takes

Copyright © 2004–2026, R (Chandra) Chandrasekhar

you directly from object A to object C. If you can take two connecting flights, there must be a direct flight.[1]

3. Associativity: If you chain three morphisms together ($A \to B \to C \to D$), it does not matter if we group the first two actions together or the last two actions together; the final destination remains exactly the same. This means $(A \to B) \to C = A \to (B \to C)$.[2]

The Commuting Triangle: Direct and connecting Flights

The most fundamental visual tool in Category Theory is the *commuting triangle*. It is the geometric expression of the second rule: Composition.

Imagine three objects: A, B, and C. Suppose we have a morphism f that takes us from A to B, and a second morphism g that takes us from B to C. Category Theory demands that there must exist a direct route, h, from A to C.

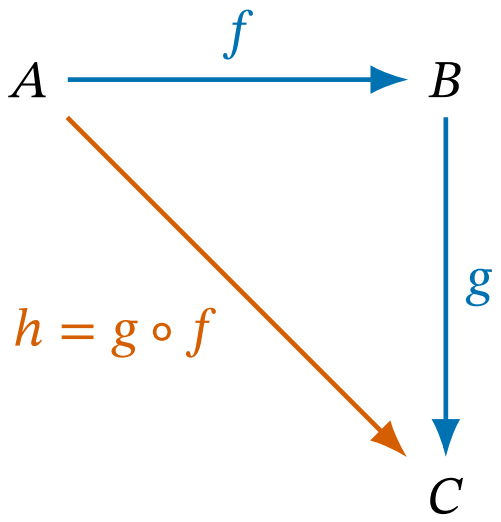

Figure 1: Diagrammatic representation of composition of morphisms. This triangle is called a *commuting triangle*. Colloquially, a direct flight h must exist which can replace two connecting flights, f and g.

Saying that this triangle "commutes" means that the path that is chosen does not matter. Taking the connecting flights—f followed by g, written as $g \circ f$—lands us at the same destination, in exactly the same state, as taking the direct flight h.[3]

This triangle requires absolutely no internal arithmetic. We are not adding or multiplying anything inside A, B, or C. This simple, unbreakable rule of *path-independence* applies universally to every single Category in the mathematical universe—from the chaotic anarchy of Sets to the rigid grids of Vector Spaces.

But what happens when our objects do have internal rules, like addition or multiplication? That is when the triangle expands into a square. We will look at this later in this blog.

If and when you meet composition in Category Theory, think of the commuting triangle, or the direct versus connecting flights analogy, and things will drop into place.

[1] You might have encountered the idea of transitivity in relations between sets, especially equivalence relations. Composition is transitivity wearing the costume of Category Theory.

[2] We bracket with parentheses because we are dealing with *binary* operations.

[3] Note the right to left association of the symbol $\circ$. $g \circ f$ means applying f first, followed by g.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Even so abstract a theory as CT does have some fundamental rules that have to be obeyed. A total absence of rules will result in an unworkable anarchy of little or no consequence.

The Category of Sets

Sets are the least rule-bound algebraic structure. They are a collection of "somethings" bound by a description. They obey logical operations like "a double negative is a positive" etc. but have no arithmetic structure like the number systems. We take a look at a simple example below.

A set supports a mapping from one object to another. Suppose there are three people and want to map them to their favourite colours. Figure 2 illustrates one such possible mapping. Note the following points:

a. No person is mapped to more than one colour. The word *favourite* excludes that kind of relationship.

b. Two people may be mapped to the same colour. No stricture prevents two people from having the same favourite colour.

c. No person is mapped to one colour. This is allowed. In other words, this is *not* disallowed.

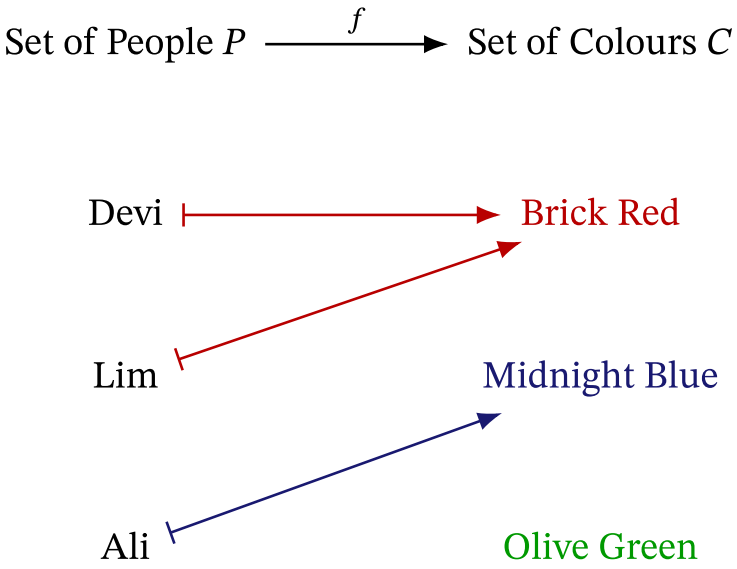

Figure 2: A mapping from a set of people to a set of their favourite colours. Note that no person is mapped to the colour Olive Green, and also that more than one person is mapped to the colour Brick Red. This illustrates the "laxness" available in set mapping.

From water to categories

Numbers and arithmetic are familiar to everyone. Calculus, differentiation, and integration to the high school student. Vector spaces and fields are the bread and butter of engineering students. We will now classify some structures in mathematics by the rigidity imposed on what they can and cannot do. What do I mean by a hierarchy of constraints that impose rigidity?

Let us use water as an example. As water vapour, it is hardly seen. It does not have dimensions, shape, etc. As a gas, it does not even have a direction of flow unless there is a wind. It simply diffuses. Water vapour resembles the elements of a set in set theory.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

When the water condenses, it might become a river with a clear direction of flow. But shape and dimensions are still elusive. Pour the water into a pitcher, and it assumes the shape of the picture. The volume may be measured.

Freeze the water in a container. The water becomes frigid and rigid. Its dimensions may be measured. It resembles a vector space. Short of melting it, the freedoms enjoyed by water in its liquid and gaseous states elude it when it is solid.

So too with mathematical structures called objects in CT. Figure 3 shows several mathematical objects, the constraints that are imposed on them, and the resulting properties they exhibit. Sets, fields, and vector spaces are likely to be more familiar than the others. This table is indicative rather than comprehensive and the purpose has been to illustrate the richness that results from following the discipline imposed on each category or space.

Category Space	Object The "Noun"	Morphism The "Verb"	The Theme Escalation of Rigidity
Sets	A raw collection of elements	Function Unrestricted mapping; complete laxness	Total anarchy. No rules, no geometry, and no arithmetic.
Topological Spaces	A space defined by "neighborhoods"	Continuous Map Bends and stretches without tearing	The faintest whisper of a rule. Measurement is impossible, but tearing is forbidden.
Metric Spaces	A space defined by strict distances	Isometry Preserves exact distances perfectly	A strict geometric rule. You are handed a ruler, and distances are absolute.
Groups	A set with a reversible operation	Homomorphism Structure-preserving mapping	A strict algebraic rule. Leaving geometry behind, every action must perfectly preserve the underlying structural symmetry.
Fields	A perfectly balanced arithmetic engine	Field Homomorphism Strictly preserves all arithmetic operations	Absolute algebraic perfection. You can add, subtract, multiply, and divide flawlessly.
Vector Spaces	A uniform grid scaled by a Field	Linear Transformation Preserves straight lines and the origin	Maximum rigidity. Strict geometry blended with strict algebra on a locked grid.

Figure 3: A table showing different mathematical objects, their properties, and the constraints governing them. Category Theory offers a neat framework for this classification.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

The Category of Groups

The "escalation of rigidity" from Sets to Groups exemplifies what happens when a strict rule is enforced in a mathematical space. In this case it is reversibility via inverses. From Figure 3 we know that groups are sets with a reversible operation where symmetry must be preserved. But what exactly do we mean by "structure-preserving mapping" or "isomorphism" in Figure 3? The defining characteristic of a group is absolute reversibility. We must be able to reverse what we have done. One consequence of this is that we must be able to return to the starting state, regardless of what it is.

Consider a collection of four numbers with multiplication, and a four-sided figure with equal sides—namely a square—undergoing rotations of 90°. Placing these two operations side-by-side, as in Figure 4, we clearly see that although we are dealing with multiplication by numbers on the one hand, and rotations of a geometric figure on the other, the similarity between the two operations are apparent even to the untutored eye.[4]

> In Group Theory, the lens of symmetry and reversibility reveals hitherto hidden similarities between objects and their actions.

$\times$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

$+$	$0°$	$90°$	$180°$	$270°$
$0°$	$0°$	$90°$	$180°$	$270°$
$90°$	$90°$	$180°$	$270°$	$0°$
$180°$	$180°$	$270°$	$0°$	$90°$
$270°$	$270°$	$0°$	$90°$	$180°$

Figure 4: The isomorphism of the C_4 group. The "C: stands for *cyclic*. On the left are repeated multiplications by the number i, and on the right, repeated rotations by 90° on the two-dimensional plane. This is what is meant by a"structure-preserving mapping".

What we have done is to abstract or "draw out" the essence of two different operations and display the resulting similarity for all to see. Like an ultraviolet light can display characters written in invisible fluorescent ink on a piece of paper, Group Theory makes obvious those patterns that might otherwise escape detection.

Once an isomorphism is identified, it might be put to work to solve a problem in a domain of mathematics that has proved difficult, but where it is either easy to solve, or already solved in another domain. This accelerates the development of mathematics as a whole, and also better integrates the various branches in it.

[4]The number i and the 90° are called *generators* of C_4 as the entire group may be generated by repeated applicatuion of the generator.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Compositions

Compositions of morphisms are a fundamental tenet of CT. They are also known in Group Theory. Let us now take a look at composition. It is better to start with an example because CT is already abstract enough as it is.

Consider these equations:

$$\begin{aligned} 2 + 3 &= 5 \\ 10^{(2+3)} &= 10^5 \\ 10^2 \times 10^3 &= 10^5 \end{aligned} \tag{1}$$

What do these equations tell us? If we take a pair of numbers, and add them, and then using 10 as base, exponentiate using the sum, we will get the same result as if we applied each number in the pair as an exponent to 10 and multiplied the result. This much is known from school algebra.

This equivalence is schematically shown in Figure 5 below, which is called a *commuting square* in the terminology of CT.

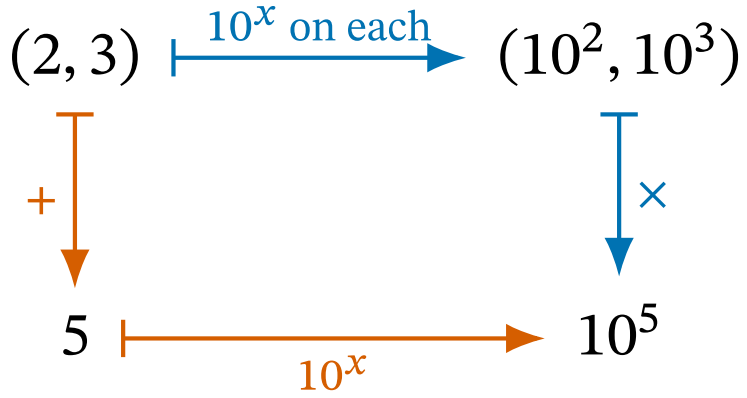

Figure 5: Pictorially, whether we move down and across or move across and then down, we get the same resut. This is in fact an example of "multiplication means addition of exponents" rule which is the basis of logarithms.

It is a concrete example of a *group isomorphism* which is a perfect structure-preserving translation between two distinct arithmetic systems: addition of real numbers and multiplication of positive real numbers.

When the above example is generalized, we get the diagram in Figure 6. We can explain this diagram using it as the template for the particular example shwon in Figure 5.

We start out with an ordered pair at the top left and end up with a single result at the bottom right. Whether we go down and right, or right and then down, we arrive at the same result.

When generalized to G and H, the visual nomenclature does not change at all; it is still called a commuting square. The only shift is in the labelling: the specific operations ($+$ and $\times$) become respectively the general symbols ($*$ and $\star$), and the specific translation rule (10^x) becomes a general morphism (ϕ).

While the diagram remains identical, Group Theory and Category Theory extract entirely different philosophical meanings from it:

Copyright © 2004–2026, R (Chandra) Chandrasekhar

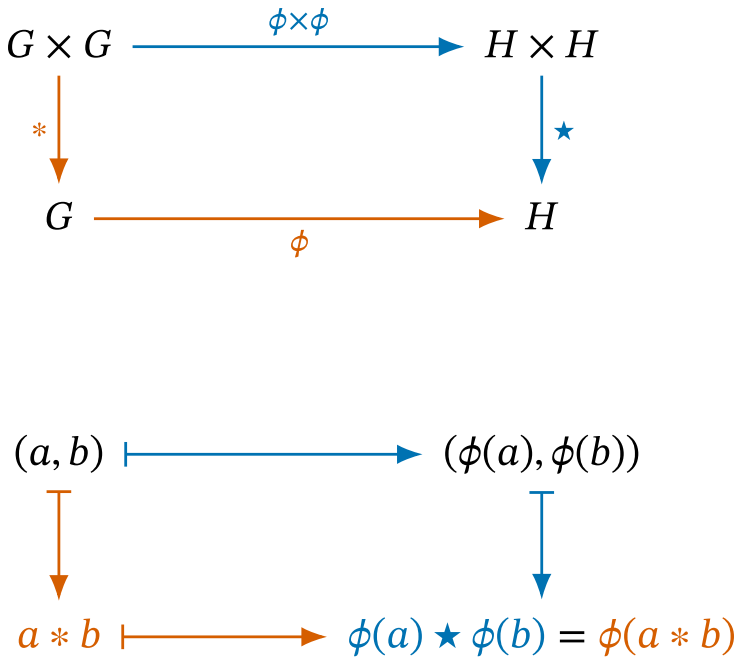

Figure 6: A commuting diagram explaining composition in Category Theory. The diagram is best analyzed in two parts. The bottom may be related to the numerical example. We then generalize from the bottom to the top.

- The Group Theory Lens (The Micro View): Group Theory cares about the *internal algebra*. It looks at the elements—like 2, 3, and 5—and verifies that the rule 10^x perfectly maps the addition of the first system to the multiplication of the second. It focuses on the specific behavior of the elements and operations to prove a "homomorphism" exists.
- The Category Theory Lens (The Macro View): Category Theory ignores the internal numbers entirely. It cares only about the *arrows* which are the morphisms. It looks at the overarching structure and verifies that the composition of the "Down-then-Right" orange arrows is mathematically identical to the composition of the "Right-then-Down" blue arrows. It is strictly concerned with *path independence*, proving that the structural integrity of the universe holds regardless of which route is travelled.

Group Theory proves that the numbers behave properly; Category Theory proves that the arrows form a closed, consistent loop.

Escalating the rigidity

Nested Russian dolls or Matryoshka dolls are extremely useful as metaphors in mathematics. I have used the idea once in another blog to help explain a nested sequence of intervals in analysis.

The escalation of rigidity in algebraic structures is another apt application of the same idea. You should be familiar with the idea os sets and subsets which are contained within sets, all the way until the empty set, $\varnothing$, which happens to be a subset of *all* sets. One may think of the empty set as being the *innermost* set within a set, without which a set cannot maintain inherence.

Likewise, in the context of a linear transformation associated with a vector space—which might be familiar to you if you are an engineering or physics student—the kernel of the transformation

Copyright © 2004–2026, R (Chandra) Chandrasekhar

represents its innermost part.

Groups are a very good example of nested algebraic structures with each inner structure obeying all the rules of the enclosing Russian doll, and in addition, some extra rules. This is shown pictorially in Figure 7.

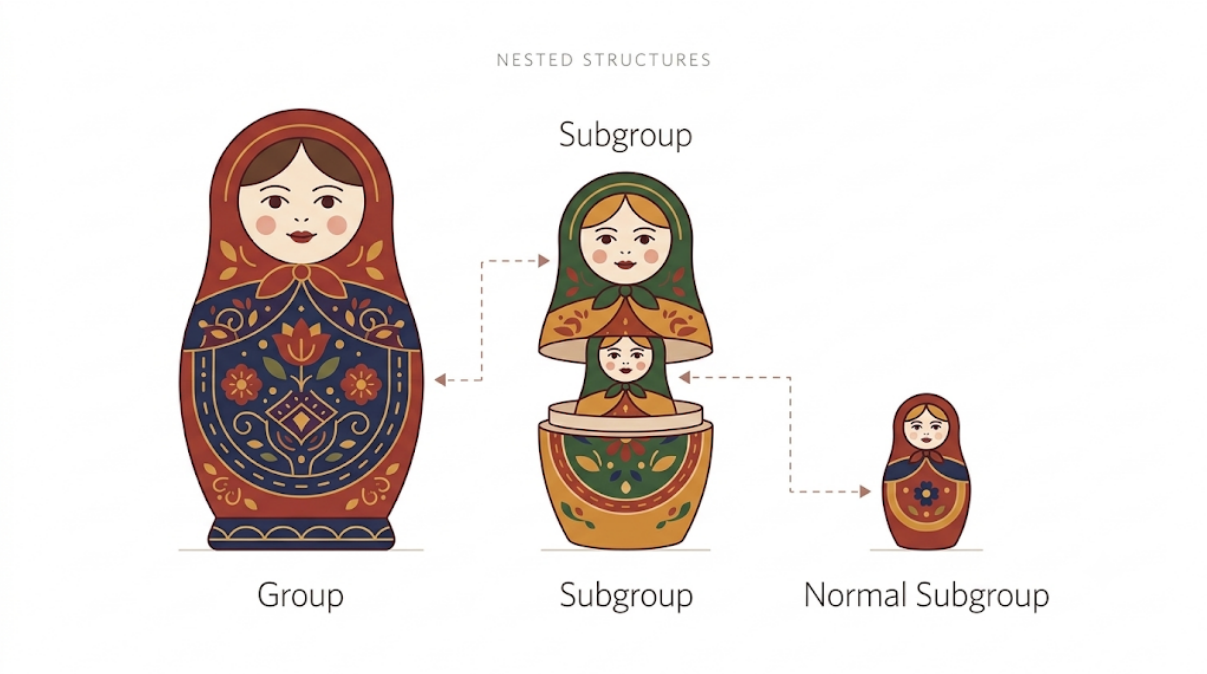

Figure 7: Nested Russin dolls ar Matryoshka dolls showing structural nesting. The deeper the nesting, the greater the constraints that apply. The innermost group is called the normal subgroup.

Likewise, groups contain subgroups, and the normal subgroup represents the innermost core—corresponding to the innermost Matryoshka doll—of the group. The deeper the nesting, the more rigid the constants that apply to the group. Like the empty set, and the kernel of a linear mapping, the normal subgroup typifies the essence of a group. Think of a seed in a fruit. The seed encapsulates the whole tree within in itself and is the most condensed representation of the tree.

Practical Applications

Groups might seem like an abstract idea suited to pure mathematics, but with little real-word application. And in this, you would be grossly mistaken. Groups are constrained by fewer rules than, say the every numbers we use. And because if this, they appear in all sorts of places as abstractions of structures.

- Particle Physics: In the prevailing paradigm, fundamental particles are not just tiny spheres; they are literally defined by their symmetries. The entire Standard Model of particle physics is built on a specific, combined mathematical group with the august. mathematical appellation $SU(3) \times SU(2) \times U(1)$. When physicists look for new particles, they are actually just looking for missing pieces in a group's symmetry table. That is how tangible something abstract has become!
- Cryptography: Every secure message, bank transfer, and encrypted website relies on finite cyclic groups. The RSA algorithm works by taking a massive number and spinning it around

Copyright © 2004–2026, R (Chandra) Chandrasekhar

a cyclic group—think of an like an enormous, multi-billion-hour clock—so many times that a hacker cannot reverse-engineer the starting position without the key. This keeps your messages and monetary transactions safe.

The Essence of Groups

It has been observed that "Numbers measure size, *groups measure symmetry*." [8]. Whether we are rotating a physical square, multiplying by imaginary numbers, or tracking the spin of a subatomic particle, we are studying the exact same phenomenon: the ability to change a system while leaving its essential core completely unchanged. Group Theory is the universal language of that preservation. It teaches us that symmetry is not just a visual property of butterfly wings and snowflakes; it is a rigid, underlying structural law of the universe.

We have now come to the end of our cursory introduction to groups. If you would like to probe further, I would suggest the reader-friendly books by Pinter [9] and Carter [10].

The Category of Fields

The word *field* is well known from its use to describe regions of space in which certain forces are experienced. Gravitational, magnetic, and electric fields are familiar examples. But the word as it is used in mathematics carries a totally different meaning.

A field is an arena in which arithmetic may be performed. Think of it as resembling a tennis court on which tennis may be played. And why do we dignify it with a dedicated name? Because fields are vital to mathematics. And because the family of fields is extensible without bound.

If a group represents the strict enforcement of the single rule of reversibility, a field represents absolute algebraic perfection. I touched upon fields in a previous blog when discussing the Fundamental Theorem of Algebra. There, it was stated that a field is a set of numbers, closed under all four basic arithmetic operations: addition, subtraction, multiplication, and division.

But how does a field fit into our Russian doll hierarchy? A field represents the progressive evolution of a group. It is in fact the child of two groups: an additive group that allows us to add and subtract, and a multiplicative group—excluding zero—that allows us to multiply and divide. It shares the same space as its two parent groups.

What happens when we "cross" these two groups? How do we derive a field? The properties of an additive group and a multiplicative group in isolation are shown in the coloured boxes in Figure 8.

But if they were simply boxed together, there would be no benefit. They could as well have remained separate. To allow the two groups to "communicate" we need to link them with a special "chromosome", or "umbilical cord", or mathematical bridge that lays out how addition and multiplication should interact to give us a consistent arithmetic.

And that is the distributive law of multiplication over addition, shown at the centre of Figure 8. This vital link breathes life into a field. The resulting happy hybrid is called a field and is fully empowered to do all of arithmetic.

Thus far, we have only concentrated on the operations or morphisms that a field is capable of. But what about the associated sets? Only certain sets of numbers qualify to be fields because they

Copyright © 2004–2026, R (Chandra) Chandrasekhar

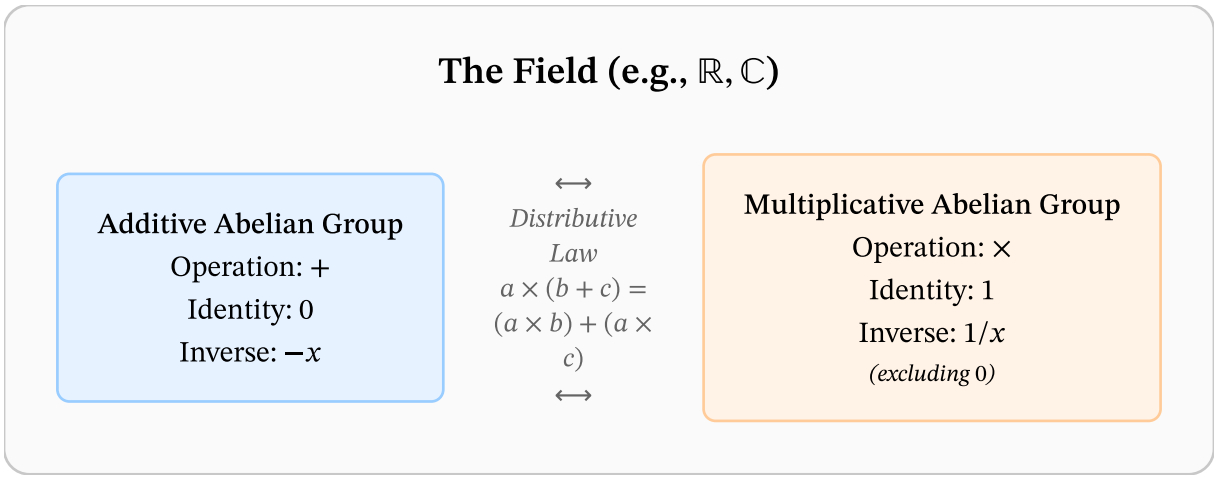

Figure 8: A field is the child of two groups connected by a special operation that makes the two parent groups function together.

must be *closed* under the arithmetic operations. Examples include the rational numbers, $\mathbb{Q}$, the real numbers, $\mathbb{R}$, and the complex numbers, $\mathbb{C}$. Two well known fields are the set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$. Are there other fields? By Jove, yes.[5]

Fields are like dance partners to vector spaces. We cannot have a vector space without naming its corresponding field. They always occur in pairs. And it is vector spaces that we turn our attention to next.

The Category of Vector Spaces

Vector Spaces are where vectors live. Vectors are pointy arrows with an attached stem.They are connected head-to-tail when added, according to the parallelogram law. They can be stretched by multiplication by a factor, say k. This scalar, k, is supplied by the field F associated with the vector space. These are the possibilities:

a. If $k < 0$, the direction of the vector is reversed.

b. If $|k| \in (0, 1)$, the size of the vector is reduced.

c. If $|k| \in (1, \infty)$, the size of the vector is increased.

d. If $k = 1$ the vector remains unchanged.

e. If $k = 0$ the vector disappears.

This much should be familiar from high school physics. But is this all there is to vector spaces? Why should they be accorded their own space?

Historically, vector spaces were identified, named, and used before the advent of Category Theory. What led to this?

Mathematicians *abstracted* from vectors their fundamental characteristics of addition with each other and multiplication by a factor. Needless to say, as in all algebraic structures, these operations are closed. They then found out that not only high school vectors, but many other mathematical

[5] They are a large family but how they became so large is a story for another blog.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

objects also possessed "vectorial characteristics". And this led to the grand generalization of the idea of a vector into that of a vector space.

Because vectors can be added, a vector space must define and enable addition of its elements. Because vectors can be scaled, there must be *some number* that can be applied to vectors to enlarge or diminish their magnitudes. This realization has two consequences:

1. Every vector space must be paired with a field that will supply the numbers to scale the vectors.
2. Addition of elements of a vector space must mirror the addition of vectors as known in high school.

Let us keep amorphous exactly what lives in a vector space. Even so, we can definitely deal with two entities: vectors, and scalars which are numbers.

Any mathematical jiggery pokery done within a vector space, using arithmetic, must satisfy the norms of vector addition and scalar multiplication. We may restrict the mappings—from one vector space to another—to functions. For historical reasons, these functions were given the moniker T rather than the usual f. Vectors are usually denoted by bold symbols and scalars by italic letters. The defining rules then become:

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \\ T(k\mathbf{u}) &= kT(\mathbf{u}). \end{aligned} \tag{2}$$

This pair of equations define the *linear transformation* which are the *only* allowed operations within a vector space.

Let us shine a bit more light on Equation (2). Let T be a map from V to W, both being vector spaces. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} = \mathbf{u} + \mathbf{v}$ all be members of V. Closure of operations in V guarantees that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all members of V.

Likewise, closure of operations in W guarantees that $T(\mathbf{u})$, $T(\mathbf{v})$, and $T(\mathbf{w})$, are all in W. Moreover, the linearity from the sum is also preserved.

If you stop to think about it, this is a strong constraint we are applying on V, W, and T. The same comments apply to the scaling part of the definition of T. The fact that Equation (2) at all holds means we are likely to reap rich dividends from this setup and, indeed, we do.

Building plan versus bricks and mortar

I want to digress briefly to introduce a metaphor about vector spaces. Apart from the rules of arithmetic encapsulated in the field F associated with a vector space, we have a set of different rules relating to vector addition. Vectors cannot, in general be multiplied. So, scalar multiplication governs the aspect of multiplication in a vector space.

We have said a fair bit about vectors, but have we pinned them down to any one instantiation, any one embodiment? If you read carefully, you will notice that we have *not particularized* the vectors, we have only *characterized* them. Vector spaces are amorphous to us as far as their contents are concerned.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Herein lies the breathtaking austerity of the linear transformation, T. It demands so little—merely that addition and scalar multiplication are respected—yet it delivers so much. Because we have strictly enforced these two rules while keeping the actual contents of the space amorphous, we are no longer restricted to vectors as lines with pointy arrows drawn on paper.

Anything that obeys these two austere rules is legally a vector. A polynomial like $3x^2 + 2x + 1$ can be added to another polynomial and scaled by a number; therefore, polynomials are vectors. A matrix, which is a grid of numbers, can be added and scaled; therefore, matrices are vectors. As we will see later, when we explore Inner Product Spaces, even continuous waves of sound can be treated as vectors.

It is this abstraction that gives vector spaces their staying power. They exist because they are useful. They are useful because they are powerful. They are powerful because the paradigm of vectors in vector spaces can be applied to very many real-life situations with exact concordance. We will see later some "far out" applications of vector spaces. For now, let us get back to the analogy.

Before a building is built, the architect and structural engineer will prepare plans the specify the way the building divides up space to perform its functions. But space cannot be divided without bricks and mortar, or plywood and drywalls. The building plan prepared by the draughtsman is the abstract vector space. The actual nuts and bolts, steel beams and floor slabs, bricks and mortar is the *manifestation* of the plan. The two are linked, one being a material realization of the other. They are in total harmony. *But they are not the same*

That same plan could be used to build a hundred apartments or houses, but they are *different* concretizations of the same plan. They are alike but not identical.

Something like this happens when we particularize a vector space by superimposing conditions on the space. Mind you these superimpositions do not alter even one jot the structure of the vector space. They are simply external appearances of the same idea. Like the fins and tail on a fish happen to aid it in swimming in water, so do our legs enable locomotion on earth.

One such grid or superimposition is to treat vectors as ordered pairs of numbers. The complex numbers may be viewed as 2D numbers. The three-dimensional Euclidean space, denoted by $\mathbb{R}^3$ may be associated with a triplet of numbers denoting a point in space with respect to a particular origin.

Note that all these are vectors, but the vector space does not worry about dimensionality or origin or any of the minutiae involved in particularizing a vector space. These attributes do not belong to the *essence* of a vector space. By abstracting the essentials from the vast multiplicity of its manifestations, we have a very concentrated "Essence of Vectors" that can be put to use anywhere vectors find application. Think of it as a juice concentrate that can be diluted.

This is the real reason why abstract algebra arose. And why Category Theory is so powerful. The "commuting square" shown below graphically portrays this power of abstraction.

The abstract vector space-to-vector space transformation gets particularized once we associate V with $\mathbb{R}^n$ and W with $\mathbb{R}^m$. The transformation T takes on the garb of matrix M. It does not matter how we get from the top left corner to the bottom right corner. We can travel across-and-down or down-and-across. The destination is the same. And this is where the commuting triangle from

Copyright © 2004–2026, R (Chandra) Chandrasekhar

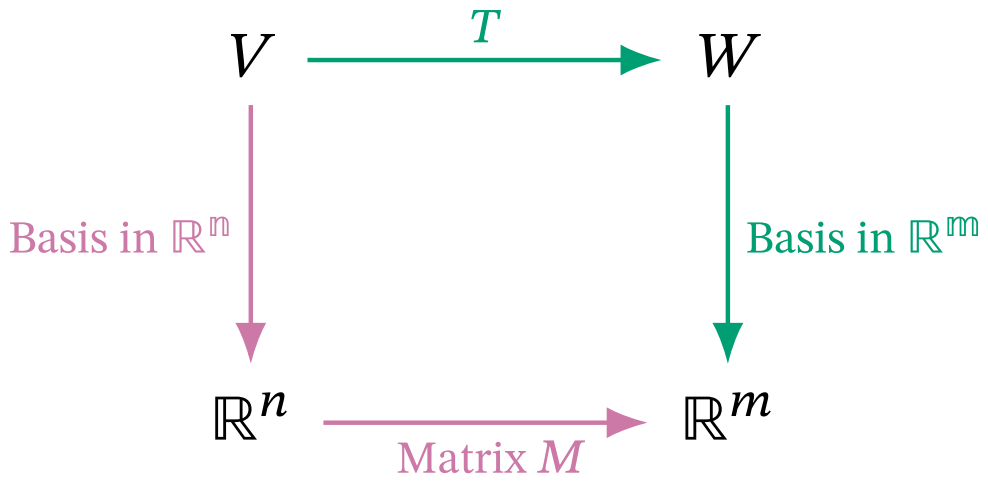

Figure 9: The vector spaces V and W are abstract. Their instantiations are $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Their bases in these vector spaces need not be the standard bases. The matrix M mirrors the transformation T and delivers the correct result in the codomain.

Figure 1 has become the commuting square in Figure 9, all guaranteed by Category Theory. Indeed, I think that Category Theory provides Abstract Algebra with guardrails.

Another subtlety is hidden in Figure 9. The upper left-to-right arrow, $V \to W$, links the coordinate-free world of pure geometry. No axes exist there. The lower left-to-right arrow, $R^n \to R^m$ is in the co-ordinate world of Descartes where geometry is cladded with the heavy numerical machinery of arithmetic and algebra. The two top-to-bottom lines bridge the abstract and the concrete. The burden of the abstract transformation T can now be assumed by the matrix M, which is capable of the heavy lifting required in the real world. We can think geometrically, but compute algebraically, without losing the truth. This connection between geometry and algebra is a recurring and powerful leitmotif in much of mathematics.

> Geometry is the vision. Algebra is the machinery.

Intermezzo: Abelian Groups and Inner Product Spaces

We will now take two short pauses in our journey to consider Abelian Groups and Inner Product Spaces, each of which holds its own place of pride in the hierarchy of algebraic structures. An analogy about how these structures evolved historically will help.

Different types of roads

Consider different types of roads. It could be a rural dirt road, a narrow village road just wide enough for two-way traffic, a city road with zebra-crossings, traffic lights, and pavements, or an eight-lane high speed expressway designed for heavy nonstop traffic travelling cross-country. All roads are meant to carry vehicles, but restrictions apply according to type and use. The highway code applies to each, but with variations suited to place, purpose, and usage.

Algebraic structures are like this. The basic ones enforce the least number of constraints because they derive their power and generality from their frugal rules. Specialized ones are built later to

Copyright © 2004–2026, R (Chandra) Chandrasekhar

serve explicit needs. That is the case with Abelian Groups vis-a-vis Groups, and Inner Product Spaces vis-a-vis Vector Spaces.

Groups and Abelian Groups

Groups were born from the tormented mind of Évariste Galois, who was perhaps one of the few mathematical revolutionaries who was also a political one. Let us cast our minds back on the groups Galois fathered.

To take the discourse on Groups one notch up the ladder of mathematical precision, let G be the set and $*$ be the operation defined on it. Let f, g, h be elements in the group. Moreover, let e be the identity element, and let the inverse of f be denoted by f^{-1}. The axioms governing Groups are four in number:

1. Closure: $(f * g) \in G \; \forall \, f, g \in G$
2. Associativity: $(f * g) * h = f * (g * h) \; \forall \, f, g, h \in G$
3. Identity element: $f * e = e * f = f \; \forall \, f \in G$
4. Inverse element: $f * f^{-1} = f^{-1} * f = e \; \forall \, f \in G$

But wait! What happened to the ability to swap the order of operations? Where is $f * g = g * f$? It has sneaked into the last two axioms, but they are special cases where the inverse and identity operations require the result to be independent of the order of the operands. But these are special cases and definitions, not general statements.

Indeed $f * g = g * f$ is *not* a group axiom. But what about the additive and multiplicative groups? They only enforce associativity, which allows a binary operation to be chained indefinitely. But there is simply no rule to enforce, say, $a + b = b + a$, or $ab = ba$. If you are shocked, you have a right to be.

In school, our first encounters with arithmetic was with the addition of numbers or the memorization of multiplication tables. These emphasize commutativity: $5 \times 2 = 2 \times 5 = 10$. The order of the operands may be swapped to give the same result. Unfortunately, this rule was not basic enough for commutativity to join the high table of the list of axioms for groups. Commutativity is not the rule of Nature—it is an incredibly rare, fragile exception, with extremely powerful consequences.

It was Niels Hendrik Abel, another gifted mathematician who also died young, who brought in the axiom of commutativity into the fold of groups. He was looking at the roots of polynomials where he felt the need for commutativity in a group. By then, the concrete floor had been laid and the plans could not be changed. So, a special type of group, later called an *Abelian Group*, was identified and named after Abel.

In the evolution of mathematics, the frugality of axioms seems to be a fundamental determinant in identifying algebraic structures. Throw almost everything away, and see if what remains can be of use. This is the guiding principle of CT that gives it the power of extreme abstraction.

Why are Abelian groups so important? The blasé answer is that they are powerful and useful. The observant among you would have noticed that I sneaked in the adjective “Abelian” in front of the additive and multiplicative groups in Figure 8, without explanation. Pardon me for that omission. It will now be redressed.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

A group becomes Abelian with the addition of a fifth axiom:

5. Commutativity: $f * g = g * f \ \forall f, g \in G$

> The commutativity of an Abelian group mathematically guarantees path independence.

Applications of Abelian Groups

Abelian groups find widespread mathematical application in contemporary mathematics. Three of these are outlined below.

a. The Foundation of Vector Spaces: Abelian groups are an indispensable prerequisite for Vector Spaces. To add vectors using the parallelogram law,[6] the order of addition cannot matter. The geometric grid must be respected.

   If we walk 3 blocks East and 4 blocks North, we must arrive at the same destination as walking 4 blocks North and 3 blocks East. If vector addition were not Abelian, physical space would warp depending on the sequence of your steps, and the entire commuting square of Category Theory would collapse. Commutativity for addition is not a luxury but a necessity in our physical world. It might be even called an attribute.

b. The Engine of Fields: Standard arithmetic takes place in fields. From Figure 8 a Field requires two Abelian groups—one for addition, one for multiplication. If you strip commutativity away, you lose standard arithmetic entirely. You would end up in the bizarre realm of matrices or quaternions, where $A \times B$ does not equal $B \times A$. Abelian groups are the only reason we can compute balance sheets, or do high school algebra. Moreover, they offer the reassurance that our earliest encounters with arithmetic were neither untrue nor in vain.

c. The Privacy Shield of the Internet: Every secure transaction on the Internet—banking, WhatsApp, email—relies on Public Key Cryptography, examples of which are Diffie-Hellman or RSA. These algorithms use incredibly large, finite Abelian groups. Because the group is Abelian, two computers can scramble a passcode at different times and mathematically guarantee that they arrive at an identical shared digital key. If the group were not commutative, secure digital communication over public channels would be mathematically impossible because temporal synchronization would be all but impossible.

Inner Product Spaces

If the signature characteristic of Abelian groups is path-independence arising from commutativity, the defining trait of Inner Product Spaces is non-interference arising from orthogonality.

The definition of the inner product is instructive in itself. Let the vector space be called V, its associated field be called F, and let $\mathbf{u}$ and $\mathbf{v}$ be *a pair of* arbitrary vectors in V.

The *inner product*, denoted by $\langle \cdot, \cdot \rangle$, is a map from the ordered pair of the two vectors to a scalar. So far, so good. But what is the definition of this map? It is not a unique map but rather one that

[6] See my blog Varieties of Multiplication if you new to this concept.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

satisfies certain conditions.[7]

To formally define this operation without destroying its generality, we must leave the function itself implicit, like a blank SCRABBLE® tile waiting to be played. The definition is:

$$\begin{aligned} &\langle\cdot,\cdot\rangle : V \times V \to F \\ &\langle\cdot,\cdot\rangle : (\mathbf{u},\mathbf{v}) \mapsto \langle\mathbf{u},\mathbf{v}\rangle, \quad \text{yielding} \\ &\langle\mathbf{u},\mathbf{v}\rangle = k \in F. \end{aligned} \tag{3}$$

Observe that lines 3 and 4 of Equation (3) together specify a map from an ordered pair of vectors to a scalar. Note also the profound asymmetry in this definition. The domain, $V \times V$, is entirely variable. V remains a formless architectural plan until the very last minute, when it is "made manifest" by a specific application. And because V is variable, the operation inside the angle brackets cannot be strictly defined in advance.

It acts like a reserve player in a game, only substituted onto the pitch "just in time". For example, if V manifests as standard 3D space, the reserve player is the algebraic dot product. If V manifests as continuous sound waves, the reserve player is a definite integral.

But look at the codomain, F. It is absolute. No matter how inanely abstract or wildly exotic the vector space V becomes, the inner product always crushes down the interaction of two vectors into a single, concrete number k residing in the familiar, underlying arithmetic field F. The domain is variable until it becomes inevitable; the codomain is the unchanging bedrock that guarantees we can actually compute a result.

Properties and Applications

Even with this degree of generality, the inner product embodies valuable properties in the results it yields.

1. Linearity:

   $$\langle(\mathbf{u}+\mathbf{v}),\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{w}\rangle + \langle\mathbf{v},\mathbf{w}\rangle \text{ and}$$
   $$\langle k\mathbf{u},\mathbf{v}\rangle = k\langle\mathbf{u},\mathbf{v}\rangle.$$

   The inner product respects the addition and scaling rules of the underlying vector space. This is no surprise.

2. Symmetry:

   $$\langle\mathbf{u},\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{u}\rangle \;\forall\, \mathbf{u},\mathbf{v} \in V$$

   The order of measurement does not matter. The shadow $\mathbf{u}$ casts on $\mathbf{v}$ is proportionally identical to the shadow $\mathbf{v}$ casts on $\mathbf{u}$. Symmetry echoes commutativity: so why do we call it by another name? This conundrum is explained later.

3. Positive Definiteness:

   $$\langle\mathbf{v},\mathbf{v}\rangle \geq 0, \text{ and}$$
   $$\langle\mathbf{v},\mathbf{v}\rangle = 0 \text{ if and only if } \mathbf{v} \text{ is the zero vector } \mathbf{0}.$$

[7] Those of you who are familiar with programming will recognize this as the difference between "pass by value" and "pass by reference".

Copyright © 2004–2026, R (Chandra) Chandrasekhar

A profound consequence of the third property is when the shadow of **u** on **v** disappears, and vice versa. This is orthogonality. But orthogonality is geometric and defines perpendicularity does it not? So, what place does it have in abstract algebra? The same ubiquitous place that the trigonometric functions occupy in much of mathematics, including in the idea of vectors.

In my blog on Varieties of Multiplication I have touched upon orthogonality. If you have studied Mechanics in high school, you will be familiar with the idea of the component of a vector in a particular direction. In the quoted blog, I have stated that "a force vector **F** has no effect perpendicular to the direction in which it acts, and this is because its component or projection in that direction is $\|\mathbf{F}\| \cos 90° = 0$."

> The sinusoids magically define mathematical non-interference through orthogonality of the inner product.

And this is the fundamental reason inner product spaces find expression as an important specialization of vector spaces. The sine and cosine functions are mutually orthogonal as are the harmonics of any frequency. By enlarging the definition of orthogonality from perpendicularity to non-interference, we enable any mathematical object that belongs in a vector space to establish the boundaries of non-interference with other vectors through the property of orthogonality.

Inner product spaces emerged out of vector spaces as a specialization because of the following reasons:

1. Vector spaces are geometrically blind: A bare vector space is austere. We can add and stretch vectors, but the space itself has no concept of length or angle. Without further rules, we cannot tell if two vectors are equal, or if they are parallel, or perpendicular, or somewhere in between.
2. Need for ruler and protractor: An inner product is an additional structural rule brought into the fray. Once equipped with this generalized dot product, the space instantly gains geometry that can be measured.[8]
3. Orthogonality: This is the "cherry on top of the whipped cream". When the inner product of two vectors evaluates to exactly zero, they cast no mathematical "shadow" on each another. They are completely independent. Orthogonality is an all-or-nothing phenomenon; there is no in-between state.

This non-interference is a powerful and universal concept. It is the single most valuable property of an inner product space, and the basis of its countless contemporary applications, of which, here are three:

a. Classical Physics: Gravity pulls strictly downwards. If we slide a crate horizontally, gravity does exactly zero work on it because the force and displacement vectors are orthogonal. The dimensions of the universe do not bleed into each other.
b. Signal Processing: Continuous waves are mathematical vectors—obedient citizens of a vector space. A 50 Hz sine wave and a 60 Hz sine wave are orthogonal in this space. This means that

[8] Again, see my earlier blog Varieties of Multiplication.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

their inner product—the integral of their product over time—is zero.[9] This mathematical truth is the basis of Fourier analysis.

c. Telecommunications: Modern cellular networks assign massive, mutually orthogonal vector codes to individual phone calls. When a cell tower computes the inner product of the incoming radio wash against a specific code, every other voice drops mathematically to zero, leaving only the desired data. Orthogonal Frequency Division Multiplexing (OFDM) is an example of this type of frugal use of bandwidth.

Untying knots of potential confusion

As we venture deeper into abstract algebra, we must confront a few deep-seated cognitive knots. Much of our mathematical conditioning from school—such as the rigid rules of BODMAS (Brackets, Orders, Division/Multiplication, Addition/Subtraction)—does collateral damage to our philosophical understanding of how mathematics actually models reality. To truly appreciate Category Theory and Group Theory, we must untangle a few vital concepts.

A Drama of Verbs, Not Characters

When we look at a set G and an operation $*$ in Group Theory, or the objects and arrows in Category Theory, a natural question arises: *If these mathematical structures are mostly about the actions (the verbs), where are the nouns? What are these actions being performed upon?*

The grand plot twist of modern mathematics is that the characters are completely interchangeable; the entire drama is driven strictly by the verbs.

In Group Theory, the elements themselves are often the actions, like "rotate 90 degrees". The "noun"—whether it is a physical Rubik's Cube, a quantum particle, or a digital passcode—is intentionally left completely unspecified. Mathematics treats the objects as generic pronouns ("it") instead of concrete nouns, focusing exclusively on the rules of the actions themselves: *Can they be chained? Can they be reversed?*

This is the mathematical equivalent of "lazy evaluation" in computer science. Just as a programming language might build up a massive chain of pending operations and refuse to compute a final value until absolutely necessary, abstract algebra builds the entire architecture of the verbs and only evaluates the nouns when forced to. We do not name the characters until they step onto the stage; we just choreograph the dance.

The Illusion of BODMAS: Associativity vs. Commutativity

Perhaps the most difficult knot to untie is the distinction between associativity and commutativity. In school, we are taught to calculate using BODMAS. This strictly conditions us to treat brackets as a temporal clock: *"do this first, and then do that."*

[9] Mathematically, we need to define the integral of their product over an interval of time T that is the least number of whole cycles for each frequency. This inner product is defined as:

$$\int_0^T \sin(100\pi t)\sin(120\pi t) = 0.$$

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Because the numbers we use in daily life obey both the associativity and commutativity rules flawlessly, we are lulled into the illusion that "grouping things" (associativity) and "swapping things" (commutativity) are part of the same mundane package. They are not. One is a matter of packaging; the other is a matter of time travel.

Associativity: The Fences of the Factory Because our mathematical operators are strictly binary, they can only digest two inputs at a time. We cannot compute $A * B * C$ in a single stroke; we must fuse them in pairs.

Associativity—written as $(A * B) * C = A * (B * C)$—guarantees that as long as the relative, chronological sequence of events is preserved, the *packaging* of those events does not change the outcome.

Consider a three-step manufacturing process:
(A) Shear the sheep,
(B) Spin the yarn, and
(C) Knit the sweater.

The physical timeline is strictly $A \to B \to C$. The brackets below merely dictate how we organize our factories:

- $(A * B) * C$: Factory 1 shears the sheep and spins the yarn, then ships it to Factory 2 to be knitted.
- $A * (B * C)$: Factory 1 only shears the sheep, shipping the raw wool to Factory 2 which spins and knits it together.

The timeline never changed. The chronological sequence of the wool's transformation is identical. The brackets do not alter the arrow of time; they simply indicate which two actions the mathematician chose to fuse together first.

Commutativity: Swapping the Timeline Commutativity, written as $A * B = B * A$, is the dangerous luxury of sequence independence. It asks: *"Can I literally swap the chronological order of two actions?"* In our factory, this would be the equivalent of trying to spin the yarn *before* shearing the sheep.

> Associativity requires three arguments because it shifts a boundary. Commutativity requires only two because it is a swap.

Notice the quintessential distinguishing factor: *Commutativity only requires two arguments* because it is fundamentally a swap. You only need two items to exchange their chronological positions. *Associativity requires three arguments* because it is about shifting a boundary. To prove that a boundary can be shifted without breaking the timeline, you must have a "middleman" (the B in our equation) to demonstrate that it can fuse just as safely with the action before it as it can with the action after it.

Category Theory's *composition* is not a synonym for either of these. Composition is the raw track

Copyright © 2004–2026, R (Chandra) Chandrasekhar

upon which the verbs run, proving that a direct route exists. But it is associativity that keeps the train on a logical timeline, and commutativity—when we are lucky enough to have it—that lets us run the trains in whatever sequence we please.

The Overloaded Word: Symmetry

Mathematics occasionally suffers from a severe case of linguistic overloading. We recycle familiar words to mean entirely different things depending on the mathematical room we happen to be sitting in. Perhaps no word is more abused in this way than *symmetry*. If you are feeling flummoxed by its multiple personalities, you are not alone.

To prevent confusion as we climb higher into abstract algebra, we must unpack the four distinct ways mathematics uses this word.

Symmetry as a mirror image This is the symmetry of our childhood. It is the bilateral symmetry of a butterfly's wings, or the reflection of a mountain in a still lake. In classical geometry, an object is symmetric if you can draw a line down its centre and fold it perfectly upon itself. It is a visual, static property of a shape.

Symmetry as commutativity When we write $A * B = B * A$, the equation itself *looks* symmetrical on the page. Because of this visual balance, commutativity is often colloquially called "symmetry of operation." But conceptually, this is the symmetry of the timeline. It guarantees that the chronological sequence of two events can be perfectly swapped without altering the final state.

Symmetry as reversibility This is the profound symmetry of Group Theory. It has nothing to do with mirror images and everything to do with actions. In a group, symmetry means absolute reversibility ($f * f^{-1} = e$). Every action you take has a guaranteed counter-action that will perfectly restore the universe to its original state. A Rubik's cube possesses immense group symmetry because every twist can be mathematically undone—even though twisting the right face and then the top face is fiercely non-commutative.

Symmetry as measurement We met this in the Inner Product Space, where $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. This is relational symmetry. It dictates that the mathematical shadow $\mathbf{u}$ casts upon $\mathbf{v}$ is proportionally identical to the shadow $\mathbf{v}$ casts upon $\mathbf{u}$. It is the guarantee that measurement is independent of perspective—the distance from your house to the tropical rain forest is exactly the same as the distance from the forest back to your house.

Why Associativity is the Bedrock

Having untangled the four faces of symmetry, a startling realization emerges: the ability to casually swap the sequence of events—commutativity—is not a fundamental law of Nature. It is a fragile luxury that only exists in highly specialized spaces.

This begs a profound question: if we strip mathematics down to its foundational pillars, past the numbers and the equations, what actually holds the edifice together?

Copyright © 2004–2026, R (Chandra) Chandrasekhar

We are left with functions. A function is simply a verb—an action that transforms an input into an output. When we chain these actions together so that the output of the first becomes the input of the second, we are performing *function composition*.

Because function composition is the fundamental engine of all mathematics, its inherent properties dictate the laws of the mathematical universe. And the single, inescapable property of function composition is associativity.

Imagine a meal. There is an expression called "from soup to nuts" describing the sequence of dishes. We progress from soup to salad through the main course to dessert and nuts. This sequence is sacrosanct. But we might be served soup along with the salad. Bracketing soup with salad does not offend the correct sequence of courses. Mind you it is (soup + salad) not (salad + soup).

Commutativity, however, attempts to swap any two courses anywhere in the sequence. Imagine starting with dessert and ending with soup. You would upset your expectations, your palate, and perhas even your stomach.

This is exactly why matrix multiplication is associative but notoriously non-commutative. A matrix is not just a grid of numbers; it is the algebra of a linear transformation—a function. Matrix multiplication is literally function composition. If matrix A transforms a 2D plane into 3D space, and matrix B transforms 3D space into 5D space, they are strictly ordered. The output of A (3D) matches the input of B (3D). They are, to use an apt phrase, *commensurable* in that specific sequence.

If you attempt to commute them and apply B first, you are trying to feed a 2D plane into an input expecting 3D space. The interfaces do not fit. The operation is undefined. Even when the matrices are perfectly square and the links do fit, swapping the order of actions—like rotating a shape 90° and then reflecting it in a mirror, versus reflecting it first and then rotating it—produces an entirely different final state.

> Commutativity is a fragile luxury. Associativity is the structural bedrock of logic itself.

An overview of Abstract Algebra

Just as a dying man says the most important things first, to maximize information transfer, we have, until now, concentrated upon the most widely used algebraic structures using the framework of CT. It is now time to do a top to toe survey and draw charcoal thumbnail sketches of all of abstract algebra. The treatment will be condensed and generously illustrated with tables and diagrams.

Exorcising the ghost of jargon

Incomprehension arising from not knowing the jargon is most off-putting. Just think of a hospital emergency room and the buzz of medical terms used there routinely and you will experience the precise feeling that assails the newcomer to abstract algebra. To help overcome this, Figure 10 is concise summary of terms that is accessible ot the layperson without losing mathematical precision.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Morphism Type	Greek Root	Technical Definition	Conceptual Meaning
Homomorphism	Homo- (Same)	A map between two spaces that preserves the algebraic structure.	The baseline "verb." It translates actions in space A to actions in space B without breaking the rules.
Isomorphism	Iso- (Equal)	A bijective (perfectly reversible) homomorphism.	A perfect translation. The two spaces are structurally identical twins wearing different clothes.
Endomorphism	Endo- (Inside)	A homomorphism from a space to itself.	An internal mapping or rewiring of a space to a subset of itself.
Automorphism	Auto- (Self)	An isomorphism from a space to itself.	A perfect symmetry action. You transformed the space, but it looks entirely unchanged.
Monomorphism	Mono- (Single)	An injective (one-to-one) homomorphism.	An embedding. The first space fits perfectly inside the second space without collapsing.
Epimorphism	Epi- (Upon)	A surjective (onto) homomorphism.	A covering. The first space completely blankets every inch of the second space.

Figure 10: A tabular summary of morphisms designed to help the newcomer to abstract algebra.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

An Etymological Aside: What's in a Name?

Before we embark on our ascent from the simplest algebraic structures to the most complex, it is worth pausing to examine the names of the structures themselves.[10]] Mathematical nomenclature often seems deliberately opaque, but looking at the etymology of these words reveals exactly how the pioneers viewed these structures as they were discovering them.

Magma: The Formless Paste The word shares a root with the geological term, coming from the Greek *magma*, meaning a "thick, kneaded mass" or ointment. In the mid-twentieth century, the famously eccentric—and entirely fictional—French mathematical collective known as Nicolas Bourbaki needed a word for a set with one binary operation and absolutely no other rules. They chose *Magma* because it perfectly describes an amorphous, unstructured blob waiting for rules to cool it into a crystalline shape embodying order and structure.

Semigroup: The Half-Baked Group The prefix *semi-* means half. A semigroup is literally "half a group." It has acquired the associative backbone—the timeline is respected—but it lacks the full symmetry and reversibility required for a true group. It is a stepping stone that has not fully matured.

Monoid: The Realm of the "One" The prefix *mono-* comes from the Greek *monas*, meaning "one" or "a single unit," while the suffix *-oid* means "resembling" or "having the form of." In early mathematics, the identity element was almost universally referred to as "the unit" or "the one" (since 1 is the identity for multiplication, the default operation of early study). Therefore, a Monoid is literally "a structure characterized by having a 'One'." The moment a Semigroup gives birth to an identity element, it is christened a Monoid in honour of that exact element.

Group: The Cluster Évariste Galois coined the term, *groupe* in French, in the 1830s. He was not thinking about abstract algebra as we know it today; he was looking at permutations (swapping roots of polynomials). He noticed that certain collections of permutations always chained together to produce another permutation within the exact same collection. They clustered together, entirely closed off from the rest of the universe. He casually called this cluster a "group," and the mundane name stuck to the most profound symmetry concept in the cosmos.

Abelian Group This group is named after Niels Hendrik Abel, and we have touched upon the etymology previously in the section Groups and Abelian Groups in this blog.

Ring Coined by David Hilbert in 1892 as *Zahlring*, or "number ring". He was studying sets of algebraic integers that "circled back" on themselves when subjected to specific operations. The term was later formalized by Emmy Noether, who stripped away the specific numbers and defined the abstract axioms we use today.

Integral Domain This name is a direct homage to the integers $\mathbb{Z}$. An Integral Domain is an abstract structure that perfectly mimics the multiplicative reliability of the integers—specifically, the guarantee that you cannot multiply two non-zero numbers and magically get zero.

[10]This is the "getting to know" you party the night before the conference opens!

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Field First used by Richard Dedekind in 1871, as Körper, which translates to "body" in German, though it was translated to "Field" in English. He used it to describe a self-contained "body" of numbers that is entirely closed under all four arithmetic operations, meaning you can wander anywhere within the field without ever stepping outside its boundaries.

Ideal Coined by Ernst Kummer in the 1840s when he invented "ideal numbers" to fix the failure of unique prime factorization in certain complex number systems. Emmy Noether later abstracted this concept. In modern algebra, an Ideal acts like a mathematical "black hole" within a Ring: if you multiply any element in the Ring by an element in the Ideal, the result is sucked into the Ideal.

The Ascent of Mount Abstract Algebra

If we start with absolute, formless anarchy and slowly forge it into a rigid geometric and algebraic object by enforcing one new rule at a time, we climb a hierarchy of nested structures. Like a set of Russian Dolls, each deeper structure inherits all the rules of the larger shells that contain it, while adding a strict new constraint of its own.

In each case we try to answer three questions abiut the abstract algebraic structure:

1. Why → Need
2. What → Characteristics
3. How → Examples

Magma

Need We require an absolute base camp—the simplest possible algebraic structure. If you throw away every single rule of mathematics except the guarantee that an action actually produces a valid result without breaking the system, what is left? A Magma.

Characteristics A set equipped with a single binary operation that satisfies exactly one rule: *Closure*. If you take any two elements a and b in the set, $a * b$ must produce an answer that also lives inside the set. There is no associativity, no identity, and no inverses.

Examples Subtraction on integers ($a - b$). The result is always an integer, so it is closed. But $(5 - 3) - 2 \neq 5 - (3 - 2)$; so the timeline is broken: it is not associative. It is merely a Magma. Another example is the game of Rock-Paper-Scissors, where the operation $*$ evaluates to "who wins." Rock $*$ Paper = Paper. The result always remains within the set $\{Rock, Paper, Scissors\}$, but it obeys almost no standard arithmetic rules.

Semigroup

Need A Magma is too chaotic. We need the arrow of time to be respected so we can build reliable, logical chains of cause and effect. We need to be able to safely drop the parentheses in a long string of operations without ambiguity.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Characteristics A Magma that adds the rule of *ssociativity*. Now, $(a * b) * c = a * (b * c)$. The chronological sequence of actions is preserved, no matter how the mathematician packages them.

Examples Strings of text. If the operation is "concatenate" (jamming words together), then ("head" + "light") + "bulb" is the exact same as "head" + ("light" + "bulb"). It always yields "headlightbulb".

Monoid

Need We need the ability to "do nothing." If you are building a system of actions, you must have a baseline state of rest. In computer programming, if you want to loop through a list and accumulate a result, you need a starting value that does not inadvertently alter the total. In biological systems this is the resting state, the diastolic blood pressure. In a gearshift motor car, it is the neutral gear.

Characteristics A Monoid is a Semigroup that has an added *Identity Element*, often denoted as e. There must exist an element where $a * e = a$ and $e * a = a$.

Examples Returning to strings of text, the identity is the empty string "". ("hello" + "" = "hello"). In standard arithmetic, the whole numbers {0, 1, 2, 3...} form a Monoid under addition, where the identity element is 0. It is not yet a group because there are no negative numbers to reverse the addition.

Group

Need We need the ability to undo an action. Without reversibility, we can only move forward, eventually getting trapped in a state we cannot escape. True mathematical symmetry requires an undo button that always works.

Characteristics A Monoid adds the *Inverse Elements*. For every element a, there exists an element a^{-1} such that $a * a^{-1} = e$. Every action has a *guaranteed* counter-action that restores the universe to its original state.

Examples All integers ($\mathbb{Z}$) under addition. The inverse of 5 is -5, because $5 + (-5) = 0$. Geometrically, the Rubik's Cube is a Group. Every twist is an action, and no matter how scrambled the cube becomes, there is a sequence of inverse actions that will perfectly restore it to a solved state. In a motorcar, the reverse gear performs this function.

Abelian Group

Need We occasionally require *sequence independence*. To navigate physical geometry or do basic accounting, we need a space where the chronological order of operations can be safely ignored.

Characteristics A Group that adds *Commutativity*. Now, $a * b = b * a$.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Examples Vector addition. If you walk 3 blocks East and 4 blocks North, you arrive at the same destination as walking 4 blocks North and 3 blocks East. The geometric grid absolutely requires commutativity to remain stable.

Ring

Need A Group only gives us one tool, like addition. To build true arithmetic, we need two interacting tools: addition and multiplication. We need a system where we can scale and stretch elements, not just slide them around.

Characteristics An Abelian Group (for addition) that introduces a second operation (multiplication). This new multiplication must be a Semigroup (it is associative), and crucially, it must *distribute* multiplication over addition: $a \times (b + c) = (a \times b) + (a \times c)$.

Examples Square matrices. You can add them (Abelian Group), and you can multiply them (associative, but notoriously non-commutative). The combination of these two operations makes the set of square matrices a Ring.

Commutative Ring

Need Matrix multiplication is non-commutative. For standard numbers and polynomials, we want multiplication to be sequence-independent as well.

Characteristics A Ring where the second operation (multiplication) is now *Commutative*: $a \times b = b \times a$.

Examples The set of even integers $\{..., -4, -2, 0, 2, 4, ...\}$ under addition and multiplication. You can add them, you can multiply them in any sequence e.g., $2 \times 4 = 4 \times 2$, and multiplication distributes perfectly over addition. But it is strictly a Ring, not a Field, because you cannot divide, e.g., $2 \div 4$ is not an even integer.

Integral Domain

Need We need to prevent mathematical anomalies where multiplying two distinct "things" mysteriously yields "nothing." We need to ensure that if the final result is zero, one of our starting components was definitely zero.

Characteristics A Commutative Ring that *eliminates Zero Divisors*. If $a \times b = 0$, then either $a = 0$ or $b = 0$.

Examples All integers ($\mathbb{Z}$) under addition and multiplication. If two integers multiply to zero, one of them has to be zero.

Contrast this with matrices, where you can multiply two non-zero matrices and get a zero matrix. This is the exact reason the Integral Domain exists. In standard arithmetic, if $A \times B = 0$, A or B must be zero. Matrices break this fundamental rule of logic.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

Here is a classic example of two non-zero matrices multiplying to yield the zero matrix.

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Although neither A nor B is zero, their product AB is zero:

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Because these "zero divisors" exist, the set of square matrices is not an Integral Domain.

Field

Need We have reached the summit of arithmetic. We need absolute freedom of motion. We must be able to infinitely scale up, and infinitely scale down. We need division.

Characteristics An Integral Domain where every non-zero element has a *Multiplicative Inverse*. You can add, subtract, multiply, and divide flawlessly.

Examples The Rational numbers ($\mathbb{Q}$), Real numbers ($\mathbb{R}$), and Complex numbers ($\mathbb{C}$). These are the perfectly balanced arenas where standard algebra and calculus are born.

The complete picture

I will round off this brief survey of the major algebraic structures with a diagram.

Figure 11 shows a nested Russian Doll diagram or rectangular Venn diagram of the principal structures of Abstract Algebra. Note how increasing depth leads to increasing constraints placed on the inner structures.

"Minor" Algebraic Structures

The structures outlined so far from Magma to Field form the primary backbone of abstract algebra. However, there are a few "minor" or specialized structures that branch off this main path. We will briefly introduce the here for the sake of completeness. You can create a brief "Specialized Topographies" subsection right after the Field:

Lattice

Not a progression of arithmetic, but a geometry of order. It is a set where any two elements have a unique "greatest lower bound" and "least upper bound", like finding the lowest common ancestor in a family tree.

Module

If a Vector Space is a rigid grid scaled by a Field, a Module is a grid scaled by a Ring. Because Rings lack division, Modules can have "torsion"—meaning we can multiply a non-zero element by a

Copyright © 2004–2026, R (Chandra) Chandrasekhar

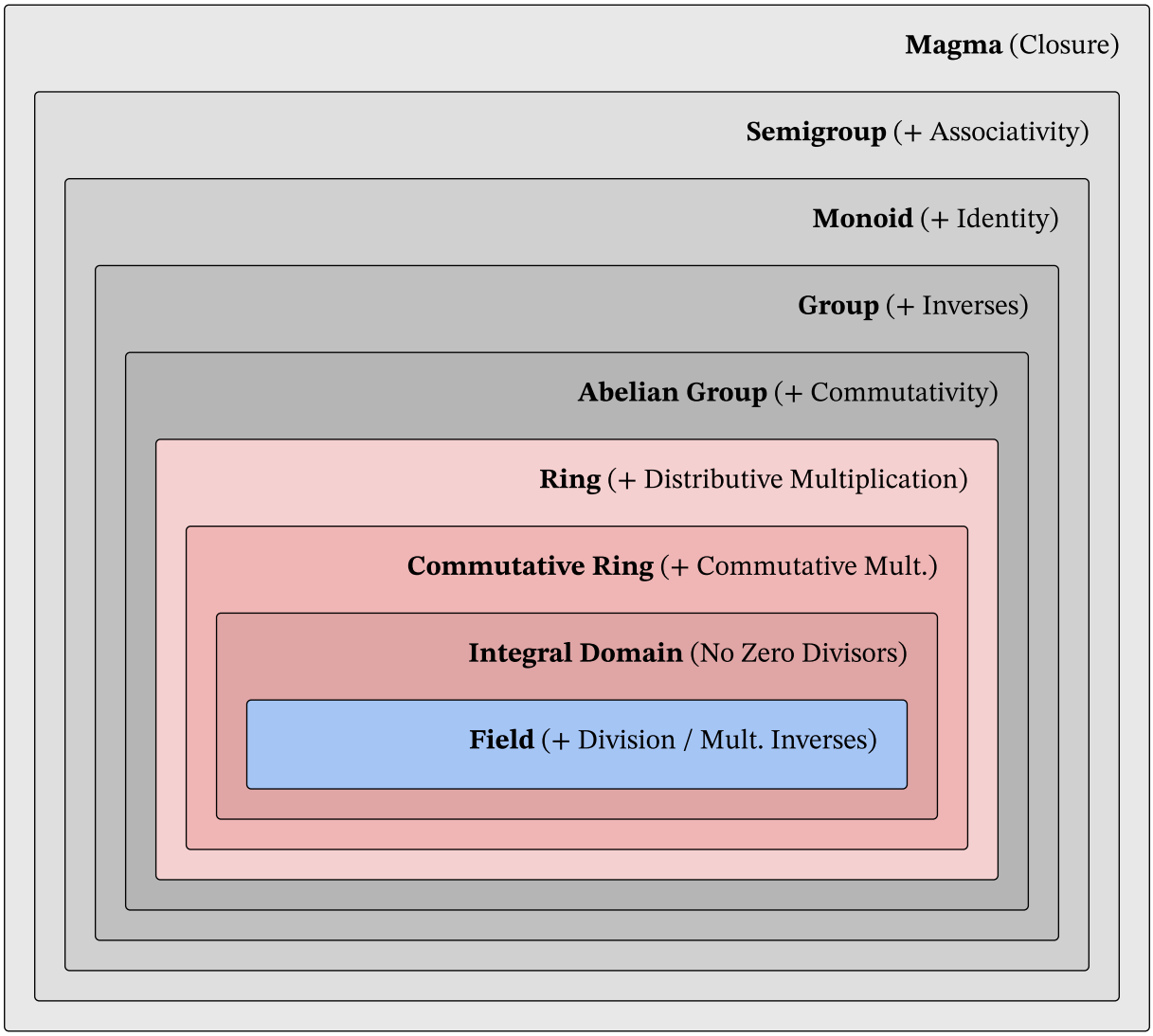

Figure 11: A nested Venn diagram or Russian doll hierarchy diagram of the principal structures of Abstract Algebra, showing the progressively stricter constraints imposed on the deeper structures.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

non-zero scalar and mysteriously arrive at zero.

Algebra (over a Field)

The ultimate hybrid. It is a Vector Space where we are also allowed to multiply the vectors themselves together in a way that respects the grid, like the cross product in 3D physics.

Objects and Connections or X and morphisms from Category Theory

Top to Bottom

Spaces

Tower diagrams

Venn diagrams

Nested rectangles/Russian dolls

The taxonomy of abstract algebra

Historical Development

Homo tables

Topy tables

Etc. tables

Links to Computer Science

"Abstract, Simplify, Generalize"

Eric S Raymond, often referred to ERS, Cathedral and the Bazaar and the Unix book. Mantra in it.

First steps to Galois Theory

Sheaves, bundles, and other arcana. Nomenclature! Nomenclature! Nomenclature!

Acknowledgements

I owe a debt of gratitude to George F Simmons, whose classic text *Introduction to Topology and Modern Analysis* pioneered the use of nested Venn diagrams to demystify the relationships between mathematical spaces. His pedagogical clarity inspired the visual approach used for the Magma-to-Field hierarchy in this essay, an example I am surprised and saddened to see so few others emulate.

Feedback

Please email me your comments and corrections.

Copyright © 2004–2026, R (Chandra) Chandrasekhar

A PDF version of this article is available for download here:

https://swanlotus.netlify.app/blogs/conceptual-abstract-algebra.pdf

References

[1] Zogg from Betelgeuse alias Martin Kuppe. Mathematics: Measuring x laziness^2. Earthlings 101, Episode 13—YouTube. Online. 1 July 2014. [Accessed 23 October 2026]. Available from: https://www.youtube.com/watch?v=XqpvBaiJRHo

[2] Marquis, Jean-Pierre. Category Theory. Stanford Encyclopedia of Philosophy. Online. 6 December 1996. [Accessed 23 October 2026]. Available from: https://plato.stanford.edu/entries/category-theory/

[3] Bradley, Tai-Danae. What is Category Theory Anyway?. Math3ma is a blog about mathematics. Online. 17 January 2017. [Accessed 23 October 2026]. Available from: https://www.math3ma.com/blog/what-is-category-theory-anyway

[4] Bradley, Tai-Danae. An invitation to category theory. Just what is category theory? Online. [Accessed 24 October 2026]. Available from: https://chalkdustmagazine.com/features/an-invitation-to-category-theory/. Chalkdust magazine, Issue 08

[5] Leinster, Tom. Basic Category Theory. Online. 26 August 2025. [Accessed 23 October 2026]. Available from: https://arxiv.org/pdf/1612.09375

[6] CauchySchwartzDaddy and others. What even is category theory anyway?. Reddit r/math. Online. 2019. [Accessed 24 October 2026]. Available from: https://www.reddit.com/r/math/comments/ft0tdw/what_even_is_category_theory_anyway/

[7] Riehl, Emily. *Category Theory in Context*. Dover Publications, 2016. Aurora: Modern Math Originals.

[8] Armstrong, Mark A. *Groups and symmetry*. Springer-Verlag, 1988.

[9] Pinter, Charles C. *A book of abstract algebra.* 2nd ed. Dover Publications, 2010. Unabridged republication of the 1990 second edition of the work originally published in 1982 by the McGraw-Hill Publishing Company, New York.

[10] Carter, Nathan C. *Visual Group Theory*. The Mathematical Association of America, 2009. Classroom Resurce Materials

Copyright © 2004–2026, R (Chandra) Chandrasekhar