Hence

$$AR \,.\, RA' + RA^2 > AM \,.\, MA' + AM \,.\, A'K,$$

or
$$AA' \,.\, AR > AM \,.\, MK$$
$$> HM \,.\, A'M, \text{ by (1).}$$

Therefore $AA' : A'M > HM : AR,$

or
$$AB^2 : BM^2 > HM : AR,$$

i.e.
$$AR^2 : BM^2 > HM : 2AR, \text{ since } AB^2 = 2AR^2,$$
$$> HM : CF.$$

Thus, since $AR = CD$, or $CE$,

(circle on diam. $EE'$) : (circle on diam. $BB'$) $> HM : CF$.

It follows that

(the cone $FEE'$) $>$ (the cone $HBB'$),

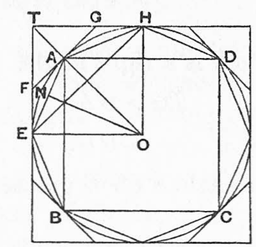
and therefore the hemisphere $DEE'$ is greater in volume than the segment $ABB'$.

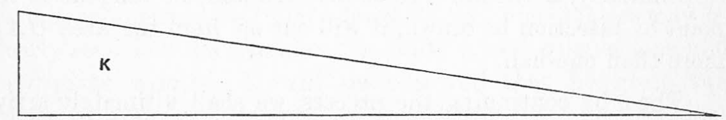
# MEASUREMENT OF A CIRCLE.

## Proposition 1.

*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*

Let $ABCD$ be the given circle, $K$ the triangle described.




Then, if the circle is not equal to $K$, it must be either greater or less.

I. If possible, let the circle be greater than $K$.

Inscribe a square $ABCD$, bisect the arcs $AB$, $BC$, $CD$, $DA$, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over $K$.

Thus the area of the polygon is greater than $K$.

Let $AE$ be any side of it, and $ON$ the perpendicular on $AE$ from the centre $O$.

Then $ON$ is less than the radius of the circle and therefore less than one of the sides about the right angle in $K$. Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in $K$.

Therefore the area of the polygon is less than $K$; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than $K$.

II. If possible, let the circle be less than $K$.

Circumscribe a square, and let two adjacent sides, touching the circle in $E$, $H$, meet in $T$. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let $A$ be the middle point of the arc $EH$, and $FAG$ the tangent at $A$.

Then the angle $TAG$ is a right angle.

Therefore $\qquad TG > GA$

$\qquad\qquad\qquad > GH$.

It follows that the triangle $FTG$ is greater than half the area $TEAH$.

Similarly, if the arc $AH$ be bisected and the tangent at the point of bisection be drawn, it will cut off from the area $GAH$ more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of $K$ over the area of the circle.

Thus the area of the polygon will be less than $K$.

Now, since the perpendicular from $O$ on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle $K$; which is impossible.



Therefore the area of the circle is not less than $K$.

Since then the area of the circle is neither greater nor less than $K$, it is equal to it.

### Proposition 2.

*The area of a circle is to the square on its diameter as* 11 *to* 14. $\pi = \frac{22}{7}$

[The text of this proposition is not satisfactory, and Archimedes cannot have placed it before Proposition 3, as the approximation depends upon the result of that proposition.]

### Proposition 3.

*The ratio of the circumference of any circle to its diameter is less than* $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.

[In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated. These various approximations and the machinery of Greek arithmetic in general will be found discussed in the Introduction, Chapter IV.]

I. Let $AB$ be the diameter of any circle, $O$ its centre, $AC$ the tangent at $A$; and let the angle $AOC$ be one-third of a right angle.

Then $OA : AC\, [= \sqrt{3} : 1] > 265 : 153$ ............ (1),

and $OC : CA\, [= 2 : 1] = 306 : 153$ .............. (2).

*First*, draw $OD$ bisecting the angle $AOC$ and meeting $AC$ in $D$.

Now $CO : OA = CD : DA$, [Eucl. VI. 3]

so that $[CO + OA : OA = CA : DA$, or]

$CO + OA : CA = OA : AD$.

Therefore [by (1) and (2)]

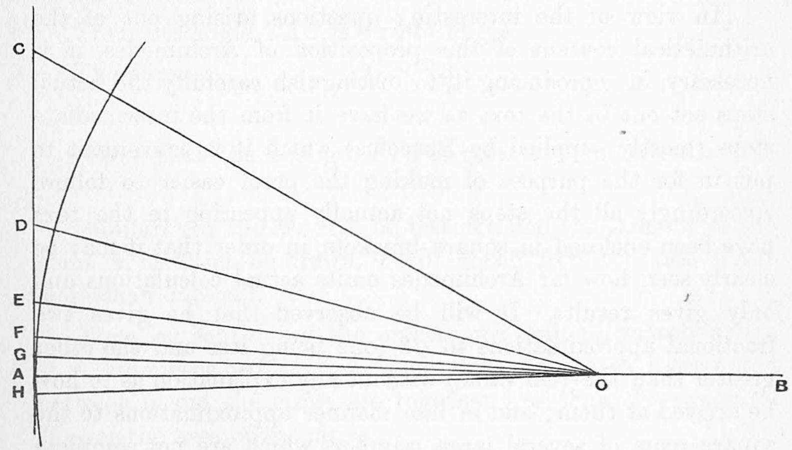
$OA : AD > 571 : 153$ ................ (3).

Hence $OD^2 : AD^2\, [= (OA^2 + AD^2) : AD^2$

$> (571^2 + 153^2) : 153^2]$

$> 349450 : 23409$,

so that $OD : DA > 591\frac{1}{8} : 153$ ................ (4).




*Secondly*, let $OE$ bisect the angle $AOD$, meeting $AD$ in $E$.

[Then $DO : OA = DE : EA$,

so that $DO + OA : DA = OA : AE$.]

Therefore $OA : AE\, [> (591\frac{1}{8} + 571) : 153$, by (3) and (4)]

$> 1162\frac{1}{8} : 153$.................... (5).



[It follows that

$OE^2 : EA^2 > \{(1162\frac{1}{8})^2 + 153^2\} : 153^2$

$> (1350534\frac{33}{64} + 23409) : 23409$

$> 1373943\frac{33}{64} : 23409$.]

Thus $OE : EA > 1172\frac{1}{8} : 153$.................... (6).

*Thirdly*, let $OF$ bisect the angle $AOE$ and meet $AE$ in $F$.

We thus obtain the result [corresponding to (3) and (5) above] that

$OA : AF\, [> (1162\frac{1}{8} + 1172\frac{1}{8}) : 153]$

$> 2334\frac{1}{4} : 153$.................... (7).

[Therefore $OF^2 : FA^2 > \{(2334\frac{1}{4})^2 + 153^2\} : 153^2$

$> 5472132\frac{1}{16} : 23409$.]

Thus $OF : FA > 2339\frac{1}{4} : 153$.................... (8).

*Fourthly*, let $OG$ bisect the angle $AOF$, meeting $AF$ in $G$.

We have then

$OA : AG\, [> (2334\frac{1}{4} + 2339\frac{1}{4}) : 153$, by means of (7) and (8)]

$> 4673\frac{1}{2} : 153$.

Now the angle $AOC$, which is one-third of a right angle, has been bisected four times, and it follows that

$\angle AOG = \frac{1}{48}$ (a right angle).

Make the angle $AOH$ on the other side of $OA$ equal to the angle $AOG$, and let $GA$ produced meet $OH$ in $H$.

Then $\angle GOH = \frac{1}{24}$ (a right angle).

Thus $GH$ is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $OA : AG > 4673\frac{1}{2} : 153$,

while $AB = 2OA, \quad GH = 2AG$,

it follows that

$AB$ : (perimeter of polygon of 96 sides) $[> 4673\frac{1}{2} : 153 \times 96]$

$> 4673\frac{1}{2} : 14688$.

But $$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$$
$$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}}\right]$$
$$< 3\tfrac{1}{7}.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter $AB$.

II. Next let $AB$ be the diameter of a circle, and let $AC$, meeting the circle in $C$, make the angle $CAB$ equal to one-third of a right angle. Join $BC$.

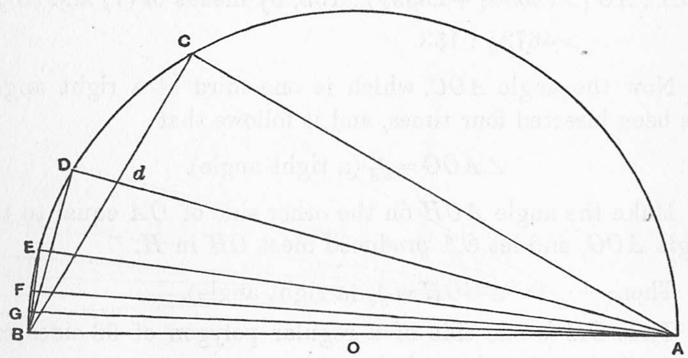
Then $\quad AC : CB \, [= \sqrt{3} : 1] < 1351 : 780.$

*First*, let $AD$ bisect the angle $BAC$ and meet $BC$ in $d$ and the circle in $D$. Join $BD$.

Then $$\angle BAD = \angle dAC = \angle dBD,$$

and the angles at $D$, $C$ are both right angles.

It follows that the triangles $ADB$, $[ACd]$, $BDd$ are similar.




Therefore $$AD : DB = BD : Dd$$
$$[= AC : Cd]$$
$$= AB : Bd \qquad \text{[Eucl. VI. 3]}$$
$$= AB + AC : Bd + Cd$$
$$= AB + AC : BC$$

or $$BA + AC : BC = AD : DB.$$



[But $\quad AC : CB < 1351 : 780$, from above,

while $\quad BA : BC = 2 : 1$
$$= 1560 : 780.]$$

Therefore $\quad AD : DB < 2911 : 780 \ldots\ldots\ldots\ldots (1).$

[Hence $\quad AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$
$$< 9082321 : 608400.]$$

Thus $\quad AB : BD < 3013\frac{3}{4} : 780 \ldots\ldots\ldots\ldots (2).$

*Secondly*, let $AE$ bisect the angle $BAD$, meeting the circle in $E$; and let $BE$ be joined.

Then we prove, in the same way as before, that
$$AE : EB \, [= BA + AD : BD$$
$$< (3013\tfrac{3}{4} + 2911) : 780, \text{ by (1) and (2)]}$$
$$< 5924\tfrac{3}{4} : 780$$
$$< 5924\tfrac{3}{4} \times \tfrac{4}{13} : 780 \times \tfrac{4}{13}$$
$$< 1823 : 240 \ldots\ldots\ldots\ldots (3).$$

[Hence $\quad AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$
$$< 3380929 : 57600.]$$

Therefore $\quad AB : BE < 1838\frac{9}{11} : 240 \ldots\ldots\ldots\ldots (4).$

*Thirdly*, let $AF$ bisect the angle $BAE$, meeting the circle in $F$.

Thus $$AF : FB \, [= BA + AE : BE$$
$$< 3661\tfrac{9}{11} : 240, \text{ by (3) and (4)]}$$
$$< 3661\tfrac{9}{11} \times \tfrac{11}{40} : 240 \times \tfrac{11}{40}$$
$$< 1007 : 66 \ldots\ldots\ldots\ldots (5).$$

[It follows that
$$AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$$
$$< 1018405 : 4356.]$$

Therefore $\quad AB : BF < 1009\frac{1}{6} : 66 \ldots\ldots\ldots\ldots (6).$

*Fourthly*, let the angle $BAF$ be bisected by $AG$ meeting the circle in $G$.

Then $$AG : GB \, [= BA + AF : BF]$$
$$< 2016\tfrac{1}{6} : 66, \text{ by (5) and (6).}$$

[And $\quad AB^2 : BG^2 < \{(2016\tfrac{1}{6})^2 + 66^2\} : 66^2$

$$< 4069284\tfrac{1}{36} : 4356.]$$

Therefore $\quad AB : BG < 2017\tfrac{1}{4} : 66,$

whence $\quad BG : AB > 66 : 2017\tfrac{1}{4}$ ......................(7).

[Now the angle $BAG$ which is the result of the fourth bisection of the angle $BAC$, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by $BG$ at the centre is

$\tfrac{1}{24}$ (a right angle).]

Therefore $BG$ is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

$$(\text{perimeter of polygon}) : AB \;[> 96 \times 66 : 2017\tfrac{1}{4}]$$
$$> 6336 : 2017\tfrac{1}{4}.$$

And $\quad \dfrac{6336}{2017\frac{1}{4}} > 3\tfrac{10}{71}.$

Much more then is the circumference of the circle greater than $3\tfrac{10}{71}$ times the diameter.

Thus the ratio of the circumference to the diameter

$$< 3\tfrac{1}{7} \text{ but } > 3\tfrac{10}{71}.$$

# ON CONOIDS AND SPHEROIDS.

## Introduction*.

"ARCHIMEDES to Dositheus greeting.

In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent you before, and also of some others discovered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty. And this is why even the propositions themselves were not published with the rest. But afterwards, when I had studied them with greater care, I discovered what I had failed in before.

Now the remainder of the earlier theorems were propositions concerning the right-angled conoid [paraboloid of revolution]; but the discoveries which I have now added relate to an obtuse-angled conoid [hyperboloid of revolution] and to spheroidal figures, some of which I call *oblong* (παραμάκεα) and others *flat* (ἐπιπλατέα).

I. Concerning the *right-angled conoid* it was laid down that, if a section of a right-angled cone [a parabola] be made to revolve about the diameter [axis] which remains fixed and

* The whole of this introductory matter, including the definitions, is translated literally from the Greek text in order that the terminology of Archimedes may be faithfully represented. When this has once been set out, nothing will be lost by returning to modern phraseology and notation. These will accordingly be employed, as usual, when we come to the actual propositions of the treatise.